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LETTER TO THE EDITOR

New non-local symmetries with pseudopotentials

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Abstract. The concept of non-local Lie–Bäcklund symmetries can be generalized by including pseudopotentials. For the KdV, HD and AKNS equations we calculate generalized symmetries of such a kind. The solitary wave solutions are obtained through the transformation of trivial solutions.

Symmetries play an important role in the construction of solutions of nonlinear partial differential equations (PDEs) [1,2]. They transform given solutions of the PDEs to new ones. Furthermore, one can construct special solutions that are invariant under the symmetry transformations. The Noether theorem relates variational symmetries to conservation laws.

In the following we shall consider systems of evolution equations

$$u_t + K(x, t, u, u_x, \ldots, u_{x\ldots x}) = 0 \tag{1}$$

with $(x,t) \in R^2$, $u = (u_1, \ldots, u_n) \in R^n$ and $u_t = \mathrm{D}_t(u)$, $u_x = \mathrm{D}_x(u), \ldots$. D_t and D_x denote the operators of total differentiation with respect to t and x, respectively.

The generators of the classical Lie point symmetries are operators of the form

$$v = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta^{u_i}\frac{\partial}{\partial u_i} \qquad \text{or equivalently} \qquad v = (\eta^{u_i} - \xi u_{ix} + \tau K_i)\frac{\partial}{\partial u_i}$$

where ξ, τ and η^{u_i} are functions of x, t and u. The corresponding finite symmetry transformations are given as the solution of the initial value problem

$$\begin{aligned} \frac{\mathrm{d}\tilde{x}}{\mathrm{d}\varepsilon} &= \xi(\tilde{x}, \tilde{t}, \tilde{u}) & \qquad \tilde{x}(\varepsilon = 0) &= x \\ \frac{\mathrm{d}\tilde{t}}{\mathrm{d}\varepsilon} &= \tau(\tilde{x}, \tilde{t}, \tilde{u}) & \qquad \tilde{t}(\varepsilon = 0) &= t \\ \frac{\mathrm{d}\tilde{u}_i}{\mathrm{d}\varepsilon} &= \eta^{u_i}(\tilde{x}, \tilde{t}, \tilde{u}) & \qquad \tilde{u}_i(\varepsilon = 0) &= u_i . \end{aligned}$$

Any conservation law

$$\frac{\partial}{\partial t}[F_i(x, t, u, u_x, \ldots, u_{x\ldots x})] - \frac{\partial}{\partial x}[G_i(x, t, u, u_x, \ldots, u_{x\ldots x})] = 0$$

of the PDEs defines a non-local potential variable p_i through

$$p_{ix} = F_i \qquad p_{it} = G_i. \tag{2}$$

In a similar way one can introduce higher order potentials from conservation laws of the prolonged system (1) and (2).

Non-local Lie–Bäcklund operators are of the form

$$v = \eta^{u_i}(x, t, u, u_x, \ldots, u_{x\ldots x}, p)\frac{\partial}{\partial u_i}$$

where p denotes a collection of potentials. The prolonged Lie–Bäcklund operator $v_{\text{pr}} = \eta^{u_i}\frac{\partial}{\partial u_i} + \eta^{u_{ix}}\frac{\partial}{\partial u_{ix}} + \ldots + \eta^{p_i}\frac{\partial}{\partial p_i} + \ldots$ is determined from the invariance requirement of the equations $u_{ix} = \mathrm{D}_x(u_i), \ldots$ and $p_{ix} = F_i$, $p_{it} = G_i, \ldots$. In general the prolongation does not close, neither for the local nor for the non-local variables and the finite symmetry transformations cannot be calculated.

There are however some non-local Lie–Bäcklund symmetries with closed prolongation [2]. They are equivalent to Lie point symmetries $v = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta^{u_i}\frac{\partial}{\partial u_i} + \eta^{p_i}\frac{\partial}{\partial p_i}$ of the prolonged system (1) and (2), where ξ, τ, η^{u_i} and η^{p_i} are functions of x, t, u and p. In this case it is possible to find the finite symmetry transformations. The theory of local and non-local Lie–Bäcklund symmetries is also described in [3–7].

Edelen [8] and Krasil'shchik and Vinogradov [9, 10] proposed a generalization of the concept of non-local symmetries by including pseudopotentials of the PDEs (1), which we also denote by p. They are defined through the first order differential equations

$$p_{ix} = F_i(x, t, u, u_x, \ldots, u_{x\ldots x}, p)$$
$$p_{it} = G_i(x, t, u, u_x, \ldots, u_{x\ldots x}, p)$$

such that the integrability conditions $p_{ixt} - p_{itx} = \frac{\partial}{\partial t}F_i - \frac{\partial}{\partial x}G_i = 0$ are fulfilled for all solutions of (1) [11].

In this letter we calculate generalized non-local symmetries of this sort for the KdV, HD and AKNS equations. They are equivalent to Lie point symmetries of prolonged systems of PDEs. The solitary wave solutions are obtained through the transformation of trivial solutions.

The KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{3}$$

has two known hierarchies of Lie–Bäcklund symmetries

$$v_n = (\mathcal{R}^n u_x)\frac{\partial}{\partial u} \qquad \tilde{v}_n = [\mathcal{R}^n(6tu_x - 1)]\frac{\partial}{\partial u} \qquad (n = 0, 1, \ldots)$$

where $\mathcal{R} = \mathrm{D}_x^2 + 4u + 2u_x\mathrm{D}_x^{-1}$ is the recursion operator. The symmetries v_n are local, whereas the symmetries $\tilde{v}_n$ are non-local for $n \geqslant 2$. The true Lie–Bäcklund symmetries do not have a closed prolongation.

The KdV equation has the well known pseudopotential p_1 with

$$p_{1x} = \lambda - u - p_1^2 \qquad p_{1t} = \frac{\partial}{\partial x}[u_x - 2(2\lambda + u)p_1] \qquad (\lambda \in R) \tag{4}$$

which is closely related to the Lax pair. The second equation of (4) is in conservation form, and we define a potential p_2 by

$$p_{2x} = p_1 \qquad p_{2t} = u_x - 2(2\lambda + u)p_1. \tag{5}$$

The inclusion of p_1 and p_2 leads to a new non-local Lie–Bäcklund symmetry $v = p_1 \exp(2p_2)\frac{\partial}{\partial u}$ of the KdV equation. The prolongation of this operator to the variable p_1 is determined from the invariance requirement of (4), that is

$$\begin{aligned} \mathrm{D}_x(\eta^{p_1}) &= -\eta^u - 2p_1\eta^{p_1} \\ \mathrm{D}_t(\eta^{p_1}) &= \mathrm{D}_x\left[\mathrm{D}_x(\eta^u) - 2p_1\eta^u - 2(2\lambda + u)\eta^{p_1}\right]. \end{aligned}$$

The solution is $\eta^{p_1} = -\frac{1}{4}\exp(2p_2)$. In order to calculate the prolongation to the variable p_2 we have to introduce another potential p_3 through

$$p_{3x} = \exp(2p_2) \qquad p_{3t} = -2(4\lambda - u - 2p_1^2)\exp(2p_2) \tag{6}$$

and we obtain $\eta^{p_2} = -\frac{1}{4}p_3$. η^{p_3} is determined by

$$\mathrm{D}_x(\eta^{p_3}) = 2\eta^{p_2}\exp(2p_2) = -\tfrac{1}{2}p_3\exp(2p_2) = -\tfrac{1}{2}p_3p_{3x}$$

and

$$\begin{aligned} \mathrm{D}_t(\eta^{p_3}) &= 2[\eta^u + 4p_1\eta^{p_1} - 2(4\lambda - u - 2p_1^2)\eta^{p_2}]\exp(2p_2) \\ &= (4\lambda - u - 2p_1^2)p_3\exp(2p_2) = -\tfrac{1}{2}p_3p_{3t}. \end{aligned}$$

Thus $\eta^{p_3} = -\frac{1}{4}p_3^2$ and the prolongation is closed.

$$v_{\mathrm{pr}} = p_1\exp(2p_2)\frac{\partial}{\partial u} - \frac{1}{4}\exp(2p_2)\frac{\partial}{\partial p_1} - \frac{1}{4}p_3\frac{\partial}{\partial p_2} - \frac{1}{4}p_3^2\frac{\partial}{\partial p_3}$$

is a Lie point symmetry for the prolonged system (3), (4), (5) and (6). The corresponding finite symmetry transformations are

$$\begin{aligned} &\tilde{x} = x \qquad \tilde{t} = t \\ &\tilde{u} = u + \frac{4\varepsilon}{4 + \varepsilon p_3}p_1\exp(2p_2) - \frac{2\varepsilon^2}{(4 + \varepsilon p_3)^2}\exp(4p_2) \\ &\tilde{p}_1 = p_1 - \frac{\varepsilon}{4 + \varepsilon p_3}\exp(2p_2) \qquad \tilde{p}_2 = p_2 + \ln\left(\frac{4}{4 + \varepsilon p_3}\right) \\ &\tilde{p}_3 = \frac{4p_3}{4 + \varepsilon p_3}. \end{aligned} \tag{7}$$

As an example we take the trivial solution $u = 0$. From (4), (5) and (6) with $\lambda > 0$ we obtain the following special solutions for the potentials:

$$\begin{aligned} &p_1 = \sqrt{\lambda} \qquad p_2 = \sqrt{\lambda}(x - 4\lambda t) \\ &p_3 = \frac{1}{2\sqrt{\lambda}}\exp[2\sqrt{\lambda}(x - 4\lambda t)]. \end{aligned}$$

Substitution into (7) leads to the transformed solution

$$\tilde{u} = 2\lambda\,\mathrm{sech}^2\left[\sqrt{\lambda}(\tilde{x} - 4\lambda\tilde{t}) + \frac{1}{2}\ln\left(\frac{\varepsilon}{8\sqrt{\lambda}}\right)\right] \qquad \varepsilon \geqslant 0$$

which is the solitary wave solution if $\varepsilon > 0$ and the trivial solution in the limit $\varepsilon \to 0$.

For the HD equation

$$u_t - u^3 u_{xxx} = 0 \tag{8}$$

we define the variables p_1 and p_2 through

$$\begin{aligned} p_{1x} &= \frac{\lambda}{u^2} - p_1^2 \qquad & p_{1t} &= 2\lambda\frac{\partial}{\partial x}(2up_1 - u_x) \\ p_{2x} &= p_1 & p_{2t} &= 2\lambda(2up_1 - u_x) \end{aligned} \qquad (\lambda \in R) \tag{9}$$

and find the symmetry $v = (u_x - 2up_1)\exp(2p_2)\frac{\partial}{\partial u}$. If we introduce another potential p_3 by

$$\begin{aligned} p_{3x} &= \lambda/u^2 \exp(2p_2) \\ p_{3t} &= -2\lambda\left(u_{xx} - 4\lambda/u - 2u_x p_1 + 2up_1^2\right)\exp(2p_2) \end{aligned} \tag{10}$$

the prolonged Lie–Bäcklund operator is equivalent to the non-projectable Lie point symmetry

$$\begin{aligned} v = {} & \exp(2p_2)\frac{\partial}{\partial x} + 2up_1\exp(2p_2)\frac{\partial}{\partial u} - p_1^2\exp(2p_2)\frac{\partial}{\partial p_1} \\ & + [p_1\exp(2p_2) - p_3]\frac{\partial}{\partial p_2} - p_3^2\frac{\partial}{\partial p_3} \end{aligned}$$

of the prolonged system (8), (9) and (10). The finite symmetry transformations are

$$\begin{aligned} \tilde{x} &= x + \varepsilon\frac{\exp(2p_2)}{1 + \varepsilon[p_3 - p_1\exp(2p_2)]} \qquad & \tilde{t} &= t \\ \tilde{u} &= u\left\{\frac{1 + \varepsilon p_3}{1 + \varepsilon[p_3 - p_1\exp(2p_2)]}\right\}^2 & \tilde{p}_1 &= p_1 - \varepsilon\frac{p_1^2\exp(2p_2)}{1 + \varepsilon p_3} \\ \tilde{p}_2 &= p_2 - \ln\{1 + \varepsilon[p_3 - p_1\exp(2p_2)]\} & \tilde{p}_3 &= \frac{p_3}{1 + \varepsilon p_3}. \end{aligned} \tag{11}$$

From the trivial solution $u = -1$, we obtain with the help of (11) for $\lambda > 0$ and

$$p_1 = \sqrt{\lambda} \qquad p_2 = \sqrt{\lambda}(x - 4\lambda t) \qquad p_3 = \frac{\sqrt{\lambda}}{2}\exp[2\sqrt{\lambda}(x - 4\lambda t)]$$

the solitary wave solution

$$\tilde{u} = \mathrm{sech}^2\left\{\sqrt{\lambda}[\tilde{x} - 4\lambda\tilde{t} + f(\tilde{x} - 4\lambda\tilde{t})] + \frac{1}{2}\ln\left(-\frac{\varepsilon\sqrt{\lambda}}{2}\right)\right\} - 1$$

where $\epsilon \leqslant 0$ and the function $f(\tilde{x}-4\lambda\tilde{t})$ is given implicitly by

$$f = \frac{1}{\sqrt{\lambda}}\left\{1 + \tanh\left[\sqrt{\lambda}(\tilde{x}-4\lambda\tilde{t}+f) + \frac{1}{2}\ln\left(-\frac{\epsilon\sqrt{\lambda}}{2}\right)\right]\right\}.$$

For the AKNS system

$$u_t - \mathrm{i}u_{xx} - 2\mathrm{i}u^2 v = 0 \qquad v_t + \mathrm{i}v_{xx} + 2\mathrm{i}uv^2 = 0. \tag{12}$$

The Lax pair gives two pseudopotentials p_1 and p_2

$$\begin{aligned} \begin{pmatrix} p_{1x} \\ p_{2x} \end{pmatrix} &= \begin{pmatrix} -\mathrm{i}\lambda & v \\ -u & \mathrm{i}\lambda \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \\ \begin{pmatrix} p_{1t} \\ p_{2t} \end{pmatrix} &= \begin{pmatrix} -\mathrm{i}uv + 2\mathrm{i}\lambda^2 & -\mathrm{i}v_x - 2\lambda v \\ -\mathrm{i}u_x + 2\lambda u & \mathrm{i}uv - 2\mathrm{i}\lambda^2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \end{aligned} \qquad (\lambda \in C) \tag{13}$$

and we find the non-local symmetry $v = p_2^2\frac{\partial}{\partial u} - p_1^2\frac{\partial}{\partial v}$. If we define another potential p_3 through

$$p_{3x} = p_1 p_2 \qquad p_{3t} = -\mathrm{i}(up_1^2 + vp_2^2 - 4\mathrm{i}\lambda p_1 p_2) \tag{14}$$

the prolonged operator

$$v_{\mathrm{pr}} = p_2^2\frac{\partial}{\partial u} - p_1^2\frac{\partial}{\partial v} - p_1 p_3\frac{\partial}{\partial p_1} - p_2 p_3\frac{\partial}{\partial p_2} - p_3^2\frac{\partial}{\partial p_3}$$

is a Lie point symmetry of the prolonged system (12), (13) and (14). The finite symmetry transformations are

$$\begin{aligned} &\tilde{x} = x \qquad \tilde{t} = t \qquad \tilde{u} = u + \epsilon\frac{p_2^2}{1+\epsilon p_3} \qquad \tilde{v} = v - \epsilon\frac{p_1^2}{1+\epsilon p_3} \\ &\tilde{p}_1 = \frac{p_1}{1+\epsilon p_3} \qquad \tilde{p}_2 = \frac{p_2}{1+\epsilon p_3} \qquad \tilde{p}_3 = \frac{p_3}{1+\epsilon p_3}. \end{aligned}$$

Let $v = -u^\star$, where the star denotes the complex conjugate. Then u satisfies the nonlinear Schrödinger NLS$^-$ equation $u_t - \mathrm{i}u_{xx} + 2\mathrm{i}u \mid u\mid^2 = 0$. For the choice $\lambda \in R$, $p_2 = p_1^\star$ and $\epsilon \in R$ we have $\tilde{v} = -\tilde{u}^\star$ and $\tilde{u}$ is again a solution of the NLS$^-$ equation. In the following we will consider the NLS$^-$ equation.

The transformation of the trivial solution $u = 0$ yields with

$$p_1 = \exp[-\mathrm{i}\lambda(x - 2\lambda t)] \qquad p_3 = x - 4\lambda t$$

the new solution

$$\tilde{u} = \epsilon\frac{\exp[2\mathrm{i}\lambda(\tilde{x} - 2\lambda\tilde{t})]}{1 + \epsilon(\tilde{x} - 4\lambda\tilde{t})}.$$

From the plane wave solution $u = \alpha\exp[\mathrm{i}(\beta x - (2\alpha^2 + \beta^2)t)]$ $(\alpha, \beta \in R)$, we obtain with $\lambda = \beta/2$ and

$$\begin{aligned} p_1 &= \exp[-\tfrac{1}{2}\mathrm{i}(\beta x - (2\alpha^2 + \beta^2)t) - \alpha(x - 2\beta t)] \\ p_3 &= -(1/2)\alpha\exp[-2\alpha(x - 2\beta t)] \end{aligned}$$

the dark soliton solution

$$\tilde{u} = \alpha \exp[\mathrm{i}(\beta\tilde{x} - (2\alpha^2 + \beta^2)\tilde{t})] \tanh \left[\alpha(\tilde{x} - 2\beta\tilde{t}) + \frac{1}{2}\ln\left(-\frac{2\alpha}{\varepsilon}\right)\right] \qquad \varepsilon \leqslant 0.$$

The inclusion of pseudopotentials leads for the KdV, HD and AKNS equations to new non-local symmetries, which are equivalent to Lie point symmetries of prolonged systems of PDEs. Analogous to the Bäcklund transformations these Lie–Bäcklund symmetries generate complex solutions, in particular the solitary wave solutions, from simple ones. The reason for this is that the transformations involve potential variables, which are determined by integration. There is however no general relationship between the known Bäcklund transformations and the given Lie–Bäcklund symmetries.

It would be interesting to investigate whether other integrable PDEs also possess symmetries of such a kind and if the inclusion of higher order potentials leads to further symmetries.

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